A Brief History of A_4 and Tribimaximal Mixing

Ernest Ma
Physics Department
University of California
Riverside, CA 92521, USA

Contents

Introduction

In 1978, soon after the putative discovery of the third family of leptons and quarks, it was conjectured by Cabibbo and Wolfenstein independently that

$$U_{l\nu}^{CW} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix},$$

where $\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$.

In 2002, Harrison/Perkins/Scott, after abandoning their bimaximal and trimaximal hypotheses, proposed the tribimaximal mixing matrix, i.e.

$$U_{l\nu}^{HPS} = \begin{pmatrix} \sqrt{2/3} & 1/\sqrt{3} & 0 \\ -1/\sqrt{6} & 1/\sqrt{3} & -1/\sqrt{2} \\ -1/\sqrt{6} & 1/\sqrt{3} & 1/\sqrt{2} \end{pmatrix} \sim (\eta_8, \eta_1, \pi^0)$$

In 2004, I was asked by Tao Han to give a plenary talk with a provocative title at the annual Pheno Symposium in Madison, so I came up with

Matrix Revelations: The Leptonic Flavor Code

without knowing at that time what I was really going to talk about. Luckily, I discovered in time the simple connection:

$$U_{l\nu}^{HPS} = (U_{l\nu}^{CW})^{\dagger} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix}.$$

This means that if $\mathcal{M}_l$ is diagonalized by $U_{l\nu}^{CW}$ and $\mathcal{M}_\nu$ has $2-3$ reflection symmetry, with zero $1-2$ and $1-3$ mixing, $U_{l\nu}^{HPS}$ will be obtained.
The first task is to find a symmetry compatible with both $U_{l\nu}^{CW}$ and (m_e, m_μ, m_τ).

Tetrahedral Symmetry A_4

For 3 families, we should look for a group with a $\underline{3}$ representation, the simplest of which is A_4, the group of the even permutation of 4 objects.

class	n	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_3
C_1	1	1	1	1	1	3
C_2	4	3	1	ω	ω^2	0
C_3	4	3	1	ω^2	ω	0
C_4	3	2	1	1	1	–1

$$\omega = \exp(2\pi i/3) = -1/2 + i\sqrt{3}/2$$

Multiplication rule:

$$\underline{3} \times \underline{3} = \underline{1}(11 + 22 + 33) + \underline{1}'(11 + \omega^2 22 + \omega 33)$$

$$+\underline{1}''(11 + \omega 22 + \omega^2 33) + \underline{3}(23, 31, 12) + \underline{3}(32, 13, 21).$$

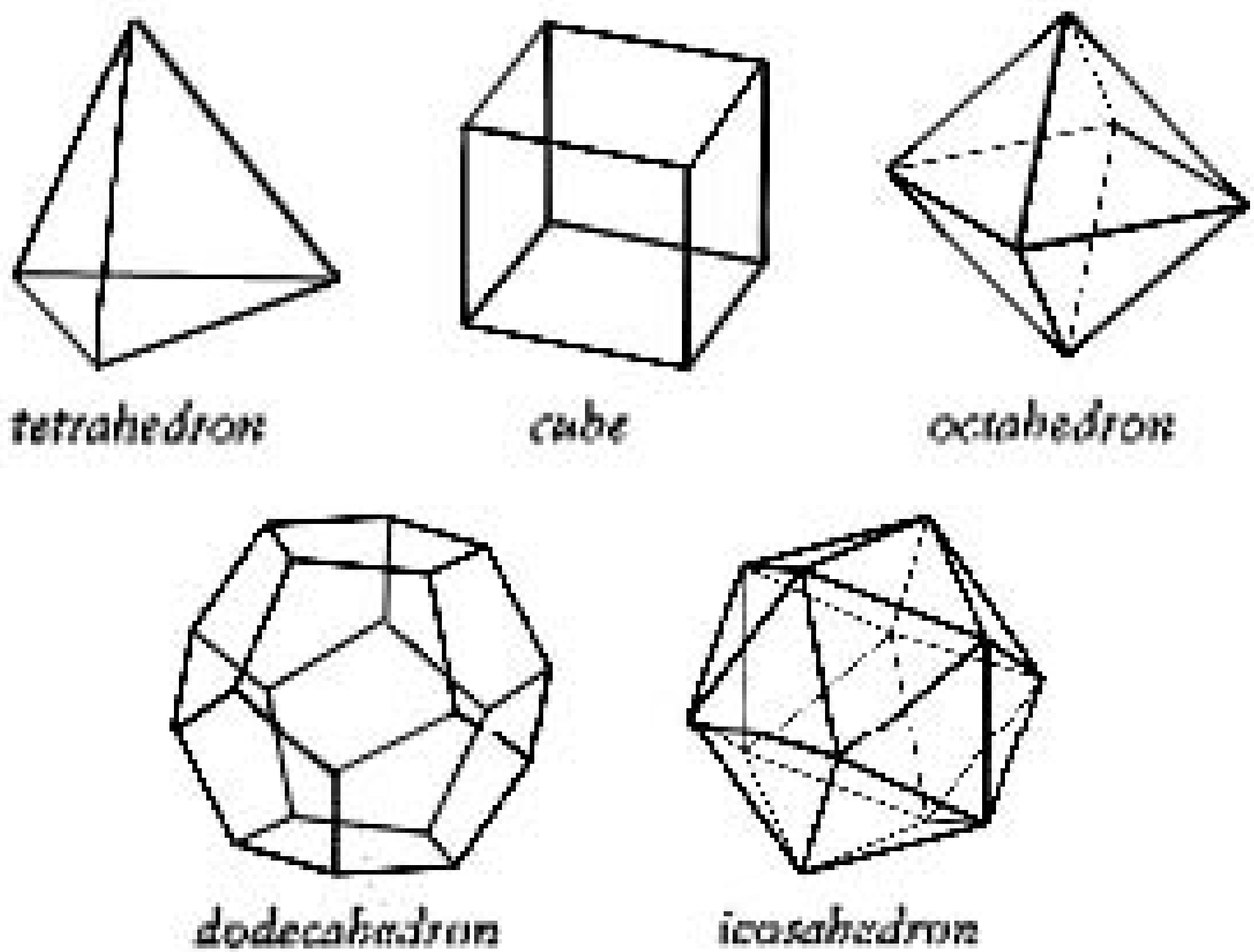
Note that $\underline{3} \times \underline{3} \times \underline{3} = \underline{1}$ is possible in A_4,
i.e. $a_1 b_2 c_3$ + permutations,
and $\underline{2} \times \underline{2} \times \underline{2} = \underline{1}$ is possible in S_3,
i.e. $a_1 b_1 c_1 + a_2 b_2 c_2$.

Perfect Three-Dimensional Geometric Solids:

solid	faces	vertices	Plato	group
tetrahedron	4	4	fire	A_4
octahedron	8	6	air	S_4
cube	6	8	earth	S_4
icosahedron	20	12	water	A_5
dodecahedron	12	20	quintessence	A_5

[Hollywood: quintessence = Milla Jovovich.]

Amusingly, there are also 5 string theories in 10 dimensions: Type I is dual to Heterotic $SO(32)$, Type IIA is dual to Heterotic $E_8 \times E_8$, and Type IIB is self-dual.


tetrahedron
cube
octahedron
dodecahedron
icosahedron


SALVADOR DALI The Sacrament of the Last Supper

Finite Subgroups of $SU(3)$:

$\Delta(3n^2)$: $\Delta(3) \equiv Z_3$, $\Delta(12) \equiv A_4$, $\Delta(27)$.

$\Delta(3n^2-3)$: $\Delta(9) \equiv Z_3 \times Z_3$, $\Delta(24) \equiv S_4$.

$SU(3)$	A_4	S_4	$\Delta(27)$
1	1	1	1_1
3	3	$3'$	3
$\bar{3}$	3	$3'$	$\bar{3}$
6	$1+1'+1''+3$	$1+2+3$	$\bar{3}+\bar{3}$
8	$1'+1''+3+3$	$2+3+3'$	$\sum_{i=2,9} 1_i$
10	$1+3+3+3$	$1'+3'+3'+3'$	$1_1 + \sum_{i=1,9} 1_i$

(I) Ma/Rajasekaran(2001): Under A_4,
$(\nu_i, l_i) \sim \underline{3}$, $l_i^c \sim \underline{1}, \underline{1}', \underline{1}''$, then with $(\phi_i^0, \phi_i^-) \sim \underline{3}$,

$$\mathcal{M}_l = \begin{pmatrix} h_1 v_1 & h_2 v_1 & h_3 v_1 \\ h_1 v_2 & h_2 \omega v_2 & h_3 \omega^2 v_2 \\ h_1 v_3 & h_2 \omega^2 v_3 & h_3 \omega v_3 \end{pmatrix}$$

$$= \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 & 0 & 0 \\ 0 & h_2 & 0 \\ 0 & 0 & h_3 \end{pmatrix} \sqrt{3} v$$

for $v_1 = v_2 = v_3 = v$. This is the starting point of most subsequent A_4 models.

(II) Ma(2006):

$(\nu_i, l_i) \sim \underline{3}$, $l_i^c \sim \underline{3}$, then with $(\phi_i^0, \phi_i^-) \sim \underline{1}, \underline{3}$,

$$\mathcal{M}_l = \begin{pmatrix} h_0 v_0 & h_1 v_3 & h_2 v_2 \\ h_2 v_3 & h_0 v_0 & h_1 v_1 \\ h_1 v_2 & h_2 v_1 & h_0 v_0 \end{pmatrix} = [\text{for } v_{1,2,3} = v]$$

$$\frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_\mu & 0 \\ 0 & 0 & m_\tau \end{pmatrix} \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega^2 & \omega \\ 1 & \omega & \omega^2 \end{pmatrix}$$

$m_e = h_0 v_0 + (h_1 + h_2) v$, $m_\mu = h_0 v_0 + (h_1 \omega + h_2 \omega^2) v$, and $m_\tau = h_0 v_0 + (h_1 \omega^2 + h_2 \omega) v$.

In either case, $U_{l\nu}^{CW}$ has been derived. To obtain $U_{l\nu}^{HPS}$, $\mathcal{M}_\nu$ must be considered. Let $\mathcal{M}_\nu$ be Majorana and come from Higgs triplets, then

$$\mathcal{M}_\nu = \begin{pmatrix} a+b+c & f & e \\ f & a+\omega b+\omega^2 c & d \\ e & d & a+\omega^2 b+\omega c \end{pmatrix}$$

where a comes from $\underline{1}$, b from $\underline{1}'$, c from $\underline{1}''$, and (d,e,f) from $\underline{3}$. To proceed further, the 6 parameters of $\mathcal{M}_\nu$ must be restricted.

Selected A_4 Models

The first two proposed A_4 models start with only $a \neq 0$, yielding thus 3 degenerate neutrino masses. In Ma/Rajasekaran(2001), the degeneracy is broken softly by $N_i N_j$ terms, allowing b, c, d, e, f to be nonzero. In Babu/Ma/Valle(2003), the degeneracy is broken radiatively through flavor-changing supersymmetric scalar lepton mass terms. In both cases, $\theta_{23} \simeq \pi/4$ is predicted. In BMV03, maximal CP violation in $U_{l\nu}$ is also predicted.

Ma(2004):

Let $b = c$ and $e = f = 0$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a+2b & 0 & 0 \\ 0 & a-b & d \\ 0 & d & a-b \end{pmatrix},$$

which is diagonalized by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/\sqrt{2} & -1/\sqrt{2} \\ 0 & 1/\sqrt{2} & 1/\sqrt{2} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & i \end{pmatrix},$$

with eigenvalues $a - b + d$, $a + 2b$, $-a + b + d$.

Thus tribimaximal mixing is achieved, which means $\tan^2\theta_{12} = 0.5$. In 2002 (when HPS proposed it), that was OK because the experimental uncertainty was large; but in 2004 (when I derived it), the value was more like 0.40 ± 0.05, so this idea did not appear to be that great. Even so, Francis Halzen told me after my Pheno talk to wait for the data to settle down. Sure enough in 2005, SNO corrected their analysis and we now have $\tan^2\theta_{12} = 0.45 \pm 0.05$, thus tribimaximal mixing is in vogue again.

However, since $\underline{1}'$ and $\underline{1}''$ are unrelated in A_4, $b = c$ is rather *ad hoc*.

Altarelli/Feruglio(2005):

Eliminate both $\underline{1}'$ and $\underline{1}''$, then $b = c = 0$ implies $m_1 = a + d$, $m_2 = a$, $m_3 = -a + d$.

Ma(2005):

$\Delta m^2_{sol} << \Delta m^2_{atm}$ implies $|d| \simeq -2|a|\cos\phi$, $|m_{1,2}|^2 \simeq |a|^2$, $|m_3|^2 \simeq |a|^2(1 + 8\cos^2\phi)$, i.e. normal ordering with

$$|m_{\nu_e}|^2 \simeq \frac{\Delta m^2_{atm}}{8\cos^2\phi} \simeq |m_{ee}|^2 + \frac{\Delta m^2_{atm}}{9}.$$

Babu/He(2005):
Let $\mathcal{M}_\nu^D \sim 1$ and

$$\mathcal{M}_N = \begin{pmatrix} A & 0 & 0 \\ 0 & A & D \\ 0 & D & A \end{pmatrix},$$

then $\mathcal{M}_\nu$ again has two parameters, with $e = f = 0$, $b = c$, $d^2 = 3b(b-a)$.
This scheme is essentially the $inverse$ of AF05, but allows both normal and inverted ordering of neutrino masses.

Technical challenge is to break A_4 spontaneously along two incompatible directions: $(1,1,1)$ and $(1,0,0)$.

Ma(2006): Add Z_3 in a supersymmetric model, with singlets carrying the A_4 symmetry at a high scale. This allows A_4 to be broken along the 2 different directions without breaking the supersymmetry.

Caveat: Although $b \neq c$ would allow $U_{e3} \neq 0$, the assumption $e = f = 0$ means that $\nu_2 = (\nu_e + \nu_\mu + \nu_\tau)/\sqrt{3}$ remains an eigenstate. The bound $|U_{e3}| < 0.16$ then implies $0.5 < \tan^2\theta_{12} < 0.52$, away from the preferred experimental value of 0.45 ± 0.05.

Singular Tribimaximal Mixing

(III) Hirsch/Ma/Valle/Villanova(2005):

Let $(\nu_i, l_i) \sim \underline{3}$, $l_i^c \sim \underline{3}$, and $(\phi_i^0, \phi_i^-) \sim \underline{1}, \underline{1}', \underline{1}''$,
then $\mathcal{M}_l$ is diagonal with

$$\begin{pmatrix} m_e \\ m_\mu \\ m_\tau \end{pmatrix} = \begin{pmatrix} 1 & 1 & 1 \\ 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \end{pmatrix} \begin{pmatrix} h_1 v_1 \\ h_2 v_2 \\ h_3 v_3 \end{pmatrix}.$$

Here $\mathcal{M}_\nu^{(e,\mu,\tau)} = \mathcal{M}_\nu$ already.

Let $d = e = f$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a+b+c & d & d \\ d & a+\omega b+\omega^2 c & d \\ d & d & a+\omega^2 b+\omega c \end{pmatrix}$$

Assume $b = c$ and rotate to the basis $[\nu_e, (\nu_\mu + \nu_\tau)/\sqrt{2}, (-\nu_\mu + \nu_\tau)/\sqrt{2}]$, then

$$\mathcal{M}_\nu = \begin{pmatrix} a+2b & \sqrt{2}d & 0 \\ \sqrt{2}d & a-b+d & 0 \\ 0 & 0 & a-b-d \end{pmatrix},$$

i.e. maximal $\nu_\mu - \nu_\tau$ mixing and $U_{e3} = 0$.

We now have $\tan 2\theta_{12} = 2\sqrt{2}d/(d-3b)$.
To obtain $\tan^2\theta_{12} = 1/2$, we need $b = 0$ (i.e. the absence of $\underline{1}'$, $\underline{1}''$) or $b = 2d/3$. However, $\Delta m^2_{sol} << \Delta m^2_{atm}$ implies $2a + b + d \to 0$, so that $\Delta m^2_{atm} \to 6bd$. This means that $\tan^2\theta_{12} = 1/2$ is a singular limit for $b = 0$ and can never be obtained. On the other hand, $b = 0$ enhances the symmetry of $\mathcal{M}_\nu$ from Z_2 to S_3, so $b \simeq 0$ is natural. Here $\tan^2\theta_{12} < 1/2$ implies inverted ordering, and $\tan^2\theta_{12} > 1/2$ implies normal ordering.
Of course, we can choose $b = 2d/3$, but it would not be a prediction.

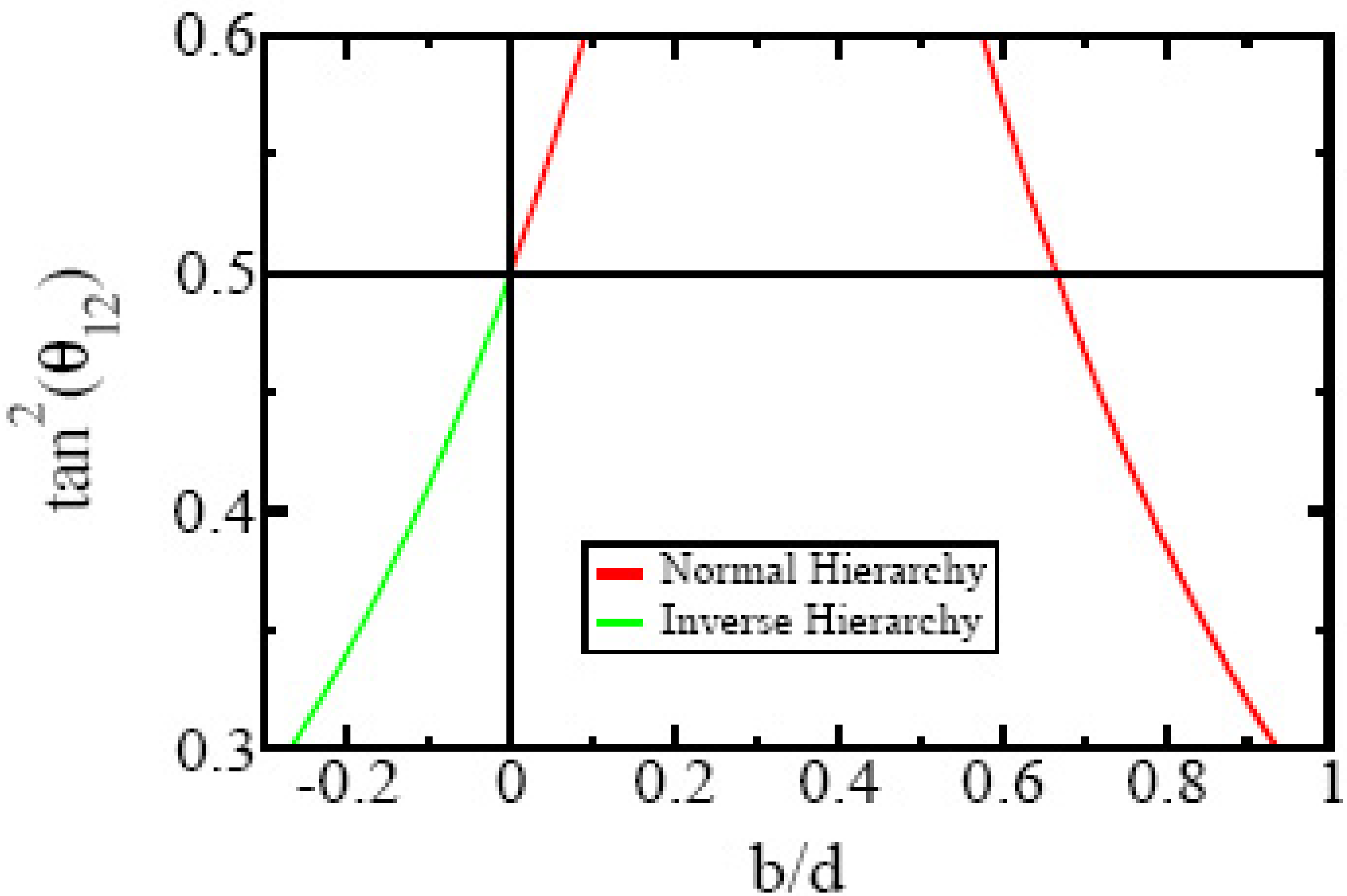


FIG. 1: Solar mixing angle, $\tan^2\theta_{12}$, vs. b/d.

Spinorial Extension of A_4

Since A_4 is a subgroup of $SO(3)$, it has a spinorial extension which is a subgroup of $SU(2)$. This is usually called the double tetrahedral group, which has 24 elements divided into 7 equivalence classes with 7 irreducible representations: $\underline{1}$, $\underline{1}'$, $\underline{1}''$, $\underline{2}$, $\underline{2}'$, $\underline{2}''$, $\underline{3}$.

Aranda/Carone/Lebed(2000)
Carr/Frampton(2007)
Feruglio/Hagedorn/Lin/Merlo(2007)
Chen/Mahanthappa(2007)

class	n	h	χ_1	$\chi_{1'}$	$\chi_{1''}$	χ_2	$\chi_{2'}$	$\chi_{2''}$	χ_3
C_1	1	1	1	1	1	2	2	2	3
C_2	4	3	1	ω	ω^2	-1	$-\omega$	$-\omega^2$	0
C_3	4	3	1	ω^2	ω	-1	$-\omega^2$	$-\omega$	0
C_4	6	4	1	1	1	0	0	0	-1
C_5	1	2	1	1	1	-2	-2	-2	3
C_6	4	6	1	ω	ω^2	1	ω	ω^2	0
C_7	4	6	1	ω^2	ω	1	ω^2	ω	0

FHLM07: $(\nu, e) \sim \underline{3}$, $e^c \sim \underline{1}, \underline{1}', \underline{1}''$, $(u, d), u^c, d^c \sim \underline{1}, \underline{2}''$.

CM07: $(\nu, e), d^c \sim \underline{3}$, $(u, d), u^c, e^c \sim \underline{1}, \underline{2}$.

Conclusion

With the application of the non-Abelian discrete symmetry A_4, a plausible theoretical understanding of the tribimaximal form of the neutrino mixing matrix has been achieved.

Other symmetries such as S_4, $\Delta(27)$, and $Sp(A_4)$ are beginning to be studied. They share some of the properties of A_4 and may help to extend our understanding of possible discrete family symmetries, with eventual links to grand unification.